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LETTER TO THE EDITOR

On the mechanical analogy of the renormalisation group in the large-*n* limit

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Abstract. Some aspects of the mechanical analogy recently proposed for the static and dynamic renormalisation group (RG) in the large-*n* limit are pointed out. In particular, starting from purely mechanical methods, a maximal set of the static RG invariants is obtained. Similar results also hold in the most complex dynamical case. Finally, by means of a canonical transformation, the original statistical mechanics problem is reduced to an equivalent anisotropic hyperbolic oscillator under stabilising conditions.

It is well known (Nicoll *et al* 1974, 1975, 1976, Nicoll and Chang 1978) that differential formulations of the Wilson RG are far more effective and convenient to apply than the finite recursion relation approaches. With this in mind, a differential formulation of the dynamic RG (DRG) in the large-*n* limit was recently presented (Busiello *et al* 1983a) for the time-dependent generalisation of the *n*-vector classical model with purely relaxational dynamics, also including the static RG (SRG) (Busiello *et al* 1981) as a particular ease.

Some peculiar aspects of the RG differential equations led us to propose (Busiello *et al* 1983a) an interesting mechanical analogy which yields an alternative geometrical picture for both the DRG and the SRG in the large-*n* limit.

Here we present some preliminary results for the dual mechanical problem which may have a relevant role in obtaining useful information about the intrinsic structure of the DRG, whose general features are less well understood (Szépfalusy and Tel 1980a, b) than those of the SRG (Ma 1973, 1976). It is suitable to start with an outline of the main ingredients leading to the mechanical analogy mentioned.

The time evolution of the order parameter $\boldsymbol{\phi}(\boldsymbol{x},\tau)\equiv\{\phi_\alpha(\boldsymbol{x},\tau);\ \alpha=1,\ldots,n\}$ of the *n*-vector classical model with purely relaxational dynamics is governed by the Langevin equation:

$$\frac{\partial\phi_\alpha}{\partial\tau}=-L\frac{\delta\mathscr{H}}{\delta\phi_\alpha}+\zeta_\alpha \tag{1}$$

where

$$\mathscr{H}\{\phi\}=\int \mathrm{d}^d x[(\nabla\boldsymbol{\phi})^2+U(\phi^2)] \tag{2}$$

and $U(\phi^2)$ is a power series of $\phi^2=\Sigma_{\alpha=1}^n\phi_\alpha^2$. In (1), $\boldsymbol{\zeta}(\boldsymbol{x},\tau)$ is a *n*-vector Gaussian white noise source and $L=\Gamma_0(\mathrm{i}\nabla)^c$ $(c=0, 2)$ where Γ_0 is a real constant, conveniently put equal to one, and $c=0$ $(c=2)$ corresponds to a non-conserved (conserved) order parameter.

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The large-n limit DRG differential equations specified for the model above are (Busiello *et al* 1983a)

$$\frac{\partial t_i}{\partial l}+(d-2)(\psi^2-F)\frac{\partial t_i}{\partial \psi^2}+(d+c)(\theta-G)\frac{\partial t_i}{\partial \theta}=a_i t_i \qquad (i=1,2) \tag{3}$$

where the t_i are unknown functions of the parameter l describing the progress of the RG averaging and of the new fields $\psi^2=\phi^2/N_c$, $\theta=[(d+c)/(d-2)](\varphi/N_c)$ with $N_c=\frac{1}{2}nK_d/(d-2)$ and $K_d=\pi^{-d/2}2^{1-d}/\Gamma(d/2)$.

Here $\varphi=\varphi(x,\tau)$ is an additional field generated by the construction of the initial large-n action within a path functional representation (Szépfalusy and Tel 1980a, b, Graham 1977, Janssen 1976, Bausch *et al* 1976) of the stochastic process (1). In (3), $a_1=2$, $a_2=4+c$ and

$$F(t_1,t_2)=[(1+t_1)^2-2t_2]^{-1/2} \qquad G(t_1,t_2)=1-(1+t_1)F(t_1,t_2). \tag{4}$$

The SRG transformation (Busiello *et al* 1981) is simply obtained setting in (4) $\theta=0$ with $t_2(l,\psi^2,0)\equiv 0$ and $t_1(l,\psi^2,0)\equiv t(l,\psi^2)=[\partial U(l,\phi^2)/\partial l]_{\phi^2=N_c\psi^2}$. With (3), to be solved with the initial conditions $t_i(0,\psi^2,\theta)=t_i^{(0)}(\psi^2,\theta)$ $(i=1,2)$, all the known results for the dynamics (Szépfalusy and Tel 1980a, b) and for the statics (Ma 1973, 1976) have been simply reproduced (Busiello *et al* 1981, 1983a) in a more natural way and some open questions have also been clarified (Busiello *et al* 1981, 1983a, Vvedensky 1984a, b). Now, we wish to emphasise a remarkable characteristic of our RG differential formulation. By inspection of (3), a very peculiar structure appears: the coefficients of the corresponding derivatives are identical. Thus they constitute a system of quasi-linear first-order partial differential equations with the 'same principal part', whose properties are well established (Courant and Hilbert 1962). This surprising feature of the DRG (and, in particular, of the SRG) transformation in the large-n limit gives the possibility of introducing a new approach to the critical dynamic (and static) problem.

In fact, the system (3) is equivalent to the single homogeneous linear partial differential equation (Busiello *et al* 1983a):

$$\partial\mathcal{S}/\partial l+H(\{q_j\};\{\partial\mathcal{S}/\partial q_j\})=0 \tag{5}$$

for an unknown function $\mathcal{S}(l,\{q_j\})$ of the five independent variables l, $q_1=\psi^2$, $q_2=t_1$, $q_3=\theta$, $q_4=t_2$ where

$$H(\{q_j\};\{p_j\})=\sum_{j=1}^{4}\alpha_j q_j p_j-[\alpha_1 F(q_2,q_4)p_1+\alpha_3 G(q_2,q_4)p_3] \tag{6}$$

is a function not depending explicitly on the parameter l, with

$$p_j=\partial\mathcal{S}/\partial q_j \qquad (j=1,\ldots,4)$$

and

$$\alpha_1=d-2 \qquad \alpha_2=2 \qquad \alpha_3=d+c \qquad \alpha_4=4+c.$$

The corresponding equations of characteristics assume the form:

$$\dot{q}_j=\partial H/\partial p_j \qquad \dot{p}_j=-\partial H/\partial q_j \qquad (j=1,\ldots,4) \tag{7}$$

where $\dot{X}=\mathrm{d}X/\mathrm{d}l$, and the integration of the original RG system of quasi-linear partial equations (3) is 'equivalent' to the integration of the ordinary differential system (7).

The results (5)–(7) provide the key for developing the mentioned mechanical analogy of the DRG (and the SRG) in the large-n limit. In fact, (5) is a Hamilton–Jacobi equation and (7) is the corresponding canonical system. Then, if one looks on the RG parameter l as a 'timelike' variable, $\mathscr{S}(l, \{q_j\})$ can be regarded as the 'action' of an 'equivalent mechanical system' whose Hamiltonian $H(\{q_j\}; \{p_j\})$ does not depend explicitly on 'time' l. Of course, since $\partial H/\partial l = \dot{H} = 0$, the 'constant of motion' H is an 'invariant' under iteration of the DRG transformation.

Notice that the mechanical analogue of the SRG is obtained from (5)-(7) setting $q_3 = q_4 = 0$ and it is defined by the action $S(l, \{q_i\}) \equiv \mathscr{S}(l, q_1, q_2, 0, 0)$ and by the Hamiltonian

$$H^{(s)}(\{q_i\}; \{p_i\}) = \sum_{i=1}^{2} \alpha_i q_i p_i - \alpha_1 p_1/(1+q_2) \tag{8}$$

with $p_i = \partial S/\partial q_i$ $(i = 1, 2)$.

New information about the original 'statistical' problem may be obtained starting from the above 'deterministic' mechanical analogy. In particular it is possible to develop a more intuitive geometrical picture of the RG in the large-n limit in the 'dual phase space'. We postpone this programme to future investigations and here we limit ourselves to exploring an important aspect of the large-n RG transformation starting from the mechanical analogy.

It is known (Nicoll *et al* 1974, 1975) that the RG invariants play an important role near criticality.

(*a*) They are strictly connected with the RG non-linear scaling fields and are generated by their appropriate combinations.

(*b*) They contain information regarding relative stability of the RG fixed points and therefore may give a global picture about criticality and crossover phenomena.

(*c*) They occur in the scaling expression of the free energy.

Furthermore, useful indications about the topology of the RG parameter space and the nature of the flux trajectories may be obtained by the knowledge of a 'maximal set' of RG invariants, i.e. the number of independent invariants characteristic of a class of models near criticality. For instance some of these invariants can be used to label the RG trajectories and may therefore be considered as a measure of the criticality of a system.

It is also established (Nicoll *et al* 1974, 1975) that the number of independent invariants is strictly related to the number of independent non-linear scaling fields. It is therefore of interest to find general procedures for the construction of RG invariants and to obtain a maximal set for both critical statics and dynamics.

Here we just show how to use the traditional tools of Hamiltonian mechanics in order to determine the invariants of the RG in the large-n limit as constants of motion in the 'dual mechanical problem'. Specifically, we do the following.

(i) Establish *a priori* the number of independent invariants characteristic of the RG in the large-n limit.

(ii) Calculate explicit expressions of invariants depending on the q and p and obtain a maximal set.

(iii) Derive invariants depending on the q only and therefore more directly connected with the original statistical problem.

Point (i) immediately follows as a direct consequence of the Hamilton–Jacobi formulation (5)-(7). Since the canonical system (7) is of order ν (equal to eight for critical dynamics and to four for statics), only $\nu - 1$ (seven and three respectively) functionally independent RG invariants exist.

Points (ii) and (iii) derive from the fact that the integrable Hamiltonian character of the dual problem allows us to use the standard construction method (Arnold 1978, Abraham and Marsden 1979, Goldstein 1980) to exhibit the whole set of constants of motion and therefore the RG invariants in the original classical statistical mechanics problem in the large-n limit. We refer, for both clarity and brevity, only to the static mechanical problem with two degrees of freedom. However, what follows can be extended, in a straightforward way, to large-n limit critical dynamics corresponding to four degrees of freedom in Hamiltonian mechanics with seven independent constants of motion.

Just by inspection, the function

$$P_1 = p_1 q_2^{(d-2)/2} \tag{9}$$

gives a constant of motion. P_1 and the static Hamiltonian $H^{(s)} = P_2$ jointly depend on q and p. A further constant of motion only depending on the q (and therefore a RG invariant depending explicitly only on the original parameters of the RG problem) can now be constructed as a conjugate variable of P_1 in a new canonical coordinate system $(Q_1, Q_2; P_1, P_2)$. The generating function $S_0 = S_0(\{q_i\}; \{P_i\})$ of the canonical transformation $(\{q_i\}; \{p_i\}) \leftrightarrow (\{Q_i\}; \{P_i\})$, which just gives a complete integral of the 'time'-independent Hamilton–Jacobi equation for $H^{(s)}$ (Busiello *et al* 1983a), satisfies the following differential relations:

$$\partial S_0/\partial q_i = p_i(\{q_j\}; \{P_j\}) \qquad \partial S_0/\partial P_i = Q_i(\{q_j\}; \{P_j\}) \tag{10}$$

with the condition

$$\det\left(\frac{\partial^2 S_0}{\partial q_i\, \partial P_j}\right) \neq 0. \tag{11}$$

Then, from (10) it follows that

$$\begin{aligned} S_0 &= \sum_{i=1}^{2} \int p_i(\{q_j\}; \{P_j\})\, \mathrm{d}q_i \\ &= \frac{P_1}{q_2^{(d-2)/2}} q_1 + \frac{P_2}{2} \ln|q_2| + \frac{d-2}{2} P_1 \int \frac{\mathrm{d}q_2}{q_2^{d/2}(1+q_2)}. \end{aligned} \tag{12}$$

Thus, the canonical map (10) is explicitly given by

$$\begin{aligned} &Q_1 = \frac{q_1}{q_2^{(d-2)/2}} + \frac{d-2}{2} \int \frac{\mathrm{d}q_2}{q_2^{d/2}(1+q_2)} \qquad Q_2 = \tfrac{1}{2} \ln|q_2| \\ &P_1 = p_1 q_2^{(d-2)/2} \qquad P_2 = (d-2) q_1 p_1 + 2 q_2 p_2 - (d-2) p_1/(1+q_2) \end{aligned} \tag{13}$$

in terms of which the equations of motion are

$$\dot{Q}_1 = 0 \qquad \dot{Q}_2 = 1 \qquad \dot{P}_1 = 0 \qquad \dot{P}_2 = 0. \tag{14}$$

Of course, the corresponding Hamiltonian is $H^{(s)} = P_2$.

The constant of motion $Q_1 = Q_1(q_1, q_2)$ is just the announced large-n limit RG invariant only depending on the q.

From our static analysis it emerges that Q_1, P_1, P_2 just constitute a maximal set of independent RG invariants. Indeed P_1 and P_2 are independent by inspection and Q_1 is independent by construction. Furthermore Q_1 is the only invariant not depending on the p since the p linearity of the Hamiltonian leads to completely separated dynamics for the q.

Analogous results can be found for the DRG but with more algebra. We omit the details and postpone to a future work a deeper discussion, together with the dual transformation also valid for the quantum RG in the large-n limit for which we have recently developed finite recursion relations and the corresponding differential formulation (Busiello *et al* 1983b).

Concerning (iii), we wish to note that the relevance, in the present context, of the constants of motion only depending on the q could suggest the use of a Lagrangian formalism. However it has to be remarked that the linearity in the p of the dual Hamiltonians (6) and (8) (and then their degeneracy in the usual mechanical sense) does not allow Legendre transformations in the given natural coordinates. Nevertheless it is possible to find new canonical coordinates removing the degeneracy of the Hamiltonian whose structure leads to a remarkable characteristic of the original statistical mechanics problem. Indeed, the canonical transformation

$$\left.\begin{aligned} q_j' &= \tfrac{1}{2} q_j - (1/\alpha_j) p_j \\ p_j' &= p_j + \tfrac{1}{2}\alpha_j q_j \end{aligned}\right\} \qquad (j=1,\ldots,4) \tag{15}$$

gives

$$H(\{q_j'\};\{p_j'\}) = H_0(\{q_j'\};\{p_j'\}) + H_1(\{q_j'\};\{p_j'\}) \tag{16}$$

where

$$H_0(\{q_j'\};\{p_j'\}) = \frac{1}{2}\sum_{j=1}^{4}(p_j'^2 - \omega_j^2 q_j'^2) \tag{17}$$

is the Hamiltonian of a four-dimensional anisotropic hyperbolic oscillator with frequencies $\omega_j = \alpha_j$ $(j=1,\ldots,4)$ and

$$H_1(\{q_j'\};\{p_j'\}) = \frac{1}{2}\sum_{K=1,3}\omega_K F_K\left(q_2' + \frac{1}{\omega_2}p_2';\ q_4' + \frac{1}{\omega_4}p_4'\right)(\omega_K q_K' - p_K') \tag{18}$$

with $F_1 = F$ and $F_3 = G$, gives rise to deviation effects from the unstable hyperbolic oscillator behaviour.

The corresponding two-dimensional representation for the SRG can be simply obtained from the previous one with $q_i' = p_i' = 0$ $(i=3, 4)$. In any case the above new coordinates are suitable to look for additional insights about the original problem in Lagrangian terms.

In conclusion, since the proposed mechanical analogy constitutes a new viewpoint in exploring the RG properties in the large-n limit, we hope that present preliminary results will stimulate further investigations on this interesting subject.

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